

### Week 5 Monday Worksheet - Improper Integral

**Instructions.** Follow the instructions of your TA and do the following problems. You are not expected to finish all the problems. So take your time! :)

1. (a)  $\int_1^\infty \frac{1}{x^p} dx$  (Consider the cases of  $p < 1$ ,  $p = 1$ ,  $p > 1$ . Here  $a$  is any positive constant.)

(b)  $\int_0^1 \frac{1}{x^p} dx$

$$\int x^{-p} dx = \begin{cases} \frac{x^{1-p}}{1-p} + C & p \neq 1 \\ \ln|x| + C & p = 1 \end{cases}$$

$$(a) ? = \lim_{M \rightarrow \infty} \int_1^M \frac{1}{x^p} dx = \begin{cases} \lim_{M \rightarrow \infty} \frac{M^{1-p} - 1}{1-p} & (\text{when } p \neq 1) \\ \lim_{M \rightarrow \infty} |\ln M| - \ln 1 & (\text{when } p = 1) \end{cases} = \begin{cases} \text{DNE} & (0 < p < 1) \\ \frac{-1}{1-p} & (p > 1) \end{cases}$$

$$(b) ? = \lim_{a \rightarrow 0} \int_a^1 \frac{1}{x^p} dx = \begin{cases} \lim_{a \rightarrow 0} \frac{1-a^{1-p}}{1-p} & (\text{when } p \neq 1) \\ \lim_{a \rightarrow 0} |\ln 1 - \ln a| & (\text{when } p = 1) \end{cases} = \begin{cases} \frac{1}{1-p} & (0 < p < 1) \\ \text{DNE} & (p > 1) \end{cases}$$

Conclusion:

$\int_1^\infty \frac{1}{x^p} dx$	Div.	Div.	<u>Conv.</u>
$\int_0^1 \frac{1}{x^p} dx$	<u>Conv.</u>	Div	Div.

2. (from 2013 fall midterm 1)  $\int_1^\infty \frac{1}{x(x^2+1)} dx$  (You may freely use the partial fraction results  $\frac{1}{x(x^2+1)} = \frac{1}{x} - \frac{x}{x^2+1}$ )

$$\int \frac{1}{x(x^2+1)} dx = \int \frac{1}{x} - \frac{x}{x^2+1} dx = \ln|x| - \frac{1}{2} \ln|x^2+1| + C = \ln\left|\frac{x}{\sqrt{x^2+1}}\right| + C$$

$$\begin{aligned} ? &= \lim_{M \rightarrow +\infty} \int_1^M \frac{1}{x(x^2+1)} dx \\ &= \lim_{M \rightarrow +\infty} \left[ \ln\left|\frac{M}{\sqrt{M^2+1}}\right| - \ln\frac{1}{\sqrt{2}} \right] \end{aligned}$$

$$= \ln 1 - \ln\frac{1}{\sqrt{2}} = -\ln\frac{1}{\sqrt{2}}$$

$$3. \int_3^\infty \frac{1}{x^2-4} dx$$

$x^2-4 = (x-2)(x+2)$

$$\frac{1}{x^2-4} = \frac{A}{x-2} + \frac{B}{x+2}$$

$$1 = A(x+2) + B(x-2)$$

$$1 = (A+B)x + 2A - 2B \Rightarrow \begin{cases} A+B=0 \\ 2A-2B=1 \end{cases}$$

$$\int \frac{1}{x^2-4} dx = \frac{1}{4} \int \frac{1}{x-2} dx - \frac{1}{4} \int \frac{1}{x+2} dx = \frac{1}{4} \ln|x-2| - \frac{1}{4} \ln|x+2| + C = \frac{1}{4} \ln \left| \frac{x-2}{x+2} \right| + C$$

$$4. \int_1^3 \frac{dt}{\sqrt{9-t^2}}$$

$$? = \lim_{M \rightarrow +\infty} \int_3^M \frac{1}{\sqrt{9-t^2}} dt = \lim_{M \rightarrow +\infty} \frac{1}{2} \ln \left| \frac{M+2}{M-2} \right| - \frac{1}{4} \ln \frac{1}{5} = \frac{1}{4} \ln 1 - \frac{1}{4} \ln \frac{1}{5} = -\frac{1}{4} \ln \frac{1}{5}$$

$$\int \frac{du}{\sqrt{1-u^2}} = \operatorname{arcsinh} u + C$$

$$\int \frac{dt}{\sqrt{9-t^2}} = \int \frac{dt}{3\sqrt{1-\frac{t^2}{9}}} = \frac{1}{3} dt \Rightarrow dt = 3du = dt$$

$$= \int \frac{3du}{3\sqrt{1-u^2}} = \operatorname{arcsinh} u + C = \operatorname{arcsinh} \frac{t}{3} + C$$

$$5. \text{ Consider the integral } \int_0^\infty \frac{\cos x}{\sqrt{1+x}} dx$$

It's ok

to ignore it → (b) Determine whether this improper integral converges or not. (No need to compute. Think of some inequalities)

for now.

$$(a) x=0 \quad \text{vertical asymptote}$$

$$x=+\infty \quad \text{infinity bound.}$$

$$(b) \int_0^\infty \frac{\cos x}{\sqrt{1+x}} dx = \int_0^\infty \frac{\cos x}{x^{\frac{1}{2}} + x^{\frac{3}{2}}} dx = \int_1^\infty \frac{\cos x}{x^{\frac{1}{2}} + x^{\frac{3}{2}}} dx + \int_0^1 \frac{\cos x}{x^{\frac{1}{2}} + x^{\frac{3}{2}}} dx$$

$$\text{Consider } \int_1^\infty \frac{\cos x}{x^{\frac{1}{2}} + x^{\frac{3}{2}}} dx$$

$$\left| \frac{\cos x}{x^{\frac{1}{2}} + x^{\frac{3}{2}}} \right| \leq \frac{1}{x^{\frac{1}{2}} + x^{\frac{3}{2}}} \leq \frac{1}{x^{\frac{3}{2}}} \Rightarrow -\frac{1}{x^{\frac{3}{2}}} \leq \frac{\cos x}{x^{\frac{1}{2}} + x^{\frac{3}{2}}} \leq \frac{1}{x^{\frac{3}{2}}}$$

$$\Rightarrow - \int_1^\infty \frac{1}{x^{\frac{3}{2}}} dx \leq \int_1^\infty \frac{\cos x}{x^{\frac{1}{2}} + x^{\frac{3}{2}}} dx \leq \int_1^\infty \frac{1}{x^{\frac{1}{2}}} dx$$

$$\text{Consider } \int_0^1 \frac{\cos x}{x^{\frac{1}{2}} + x^{\frac{3}{2}}} dx$$

$$\left| \frac{\cos x}{x^{\frac{1}{2}} + x^{\frac{3}{2}}} \right| \leq \frac{1}{x^{\frac{1}{2}} + x^{\frac{3}{2}}} \leq \frac{1}{x^{\frac{1}{2}}} \Rightarrow -\frac{1}{x^{\frac{1}{2}}} \leq \frac{\cos x}{x^{\frac{1}{2}} + x^{\frac{3}{2}}} \leq \frac{1}{x^{\frac{1}{2}}}$$

$$\Rightarrow \underbrace{- \int_0^1 \frac{1}{x^{\frac{1}{2}}} dx}_{\text{finite}} \leq \int_0^1 \frac{\cos x}{x^{\frac{1}{2}} + x^{\frac{3}{2}}} dx \leq \underbrace{\int_0^1 \frac{1}{x^{\frac{1}{2}}} dx}_{\text{finite}}$$

$$\Rightarrow \int_0^\infty \frac{\cos x}{x^{\frac{1}{2}} + x^{\frac{3}{2}}} dx = \text{"finite + finite"} \text{ is finite.}$$